Validity of the differential equations for ionization cooling

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We examine the validity of the differential equations used to describe ionization cooling. We find that the simple heating term due to multiple scattering given by D. Neuffer is a good approximation to the expression obtained from a more rigorous derivation.

Ionization cooling of the muon beam emittance is a crucial component of current designs for a muon-muon collider. The idea was first proposed by Skrinsky[1]. Neuffer[2] has continued to champion the idea and has written several articles giving practical equations for the amount of emittance cooling.

1. Neuffer's equations for ionization cooling

Neuffer gives the following expression for the decrease in the normalized emittance ε_N of a muon beam traversing matter due to energy loss

$$\frac{d\epsilon_N}{dz}(cool) = -\frac{1}{\beta^2}\frac{\epsilon_N}{E}\left|\frac{dE}{dz}\right| \tag{1}$$

where β is the usual relativistic factor, E is the total energy of the particles and dE/dz is the ionization energy loss for the particle in the material. There is also an increase in normalized emittance resulting from multiple scattering in the material. Neuffer gives a general expression for the "heating" term

$$\frac{d\epsilon_N}{dz}(heat) = \beta\gamma\frac{\beta_\perp}{2}\frac{d}{dz}<\theta^2> \tag{2}$$

where γ is the usual relativistic factor, $\beta_\perp$ is the betatron focusing parameter, θ is the angle of the particle trajectory projected onto the y-z plane, and < > represents the mean of the distribution in question over many scatters and over initial conditions. Neuffer also gives a specific formula for the heating term by substituting for the expectation value of θ

$$\frac{d\epsilon_N}{dz}(heat) = \frac{\beta_\perp}{2} \frac{E_S^2}{\beta^3 E m c^2} \frac{1}{L_R} \tag{3}$$

where E_S = 13.6 MeV, mc^2 is the particle's rest energy, and L_R is the radiation length for the scattering medium. Finally, Neuffer defines a "minimum emittance" by equating the cooling derivative in Eq. 1 with the heating derivative of Eq. 3

$$\epsilon_N = \frac{\beta_\perp}{2} \frac{E_S^2}{\beta m c^2 L_R \left|\frac{dE}{dz}\right|} \tag{4}$$

2. General derivation of ionization cooling equations

We start with the definition of normalized transverse emittance

$$\epsilon_N = \beta \gamma \epsilon \tag{5}$$

where ε is the geometric emittance. Consider the change in ε_N as the beam travels along the z direction into the material.

$$\frac{d\epsilon_N}{dz} = \epsilon \frac{d(\beta\gamma)}{dz} + \beta\gamma \frac{d\epsilon}{dz} \tag{6}$$

The first term on the right hand side represents cooling of the normalized emittance and can be written straightforwardly as the expression in Eq. 1. For our case the second term on the right hand side of Eq. 6 takes into account the increase of emittance that results from multiple scattering in the material. It is this second (heating) term, which Neuffer wrote in the form of Eq. 3, that has made some people feel uncomfortable.

In the general case the geometric emittance is defined statistically as

$$\epsilon^2 = \langle y^2 \rangle \langle \theta^2 \rangle - \langle y\,\theta \rangle^2 \tag{7}$$

where y is a dimension transverse to the particle's direction of motion, θ is the angle of the particle trajectory projected onto the y-z plane, and $\langle\ \rangle$ represents the mean of the distribution in question over many scatters and over initial conditions. The change in the geometric emittance as the beam proceeds through the material is given by

$$2\epsilon \frac{d\epsilon}{dz} = \langle y^2 \rangle \frac{d}{dz} \langle \theta^2 \rangle + \langle \theta^2 \rangle \frac{d}{dz} \langle y^2 \rangle - 2 \langle y\theta \rangle \frac{d}{dz} \langle y\theta \rangle \tag{8}$$

We note here that if we drop the second and third terms in Eq. 8 and use the relation

$$\langle y^2 \rangle = \beta_{\perp}\, \epsilon \tag{9}$$

from betatron focusing theory, the heating term can be written as the expression used by Neuffer in Eq. 2. There is a natural tendency for a beam of particles scattering their way through some material to spread out laterally and for a correlation to build up between the beam's angle and transverse position. It has been suggested by Palmer that this can be prevented by an external focusing field with sufficient strength to prevent the beams from spreading laterally. We conclude that a necessary condition for dropping the extra terms in Eq. 8, and hence for the validity of Eq. 3, is the presence in the scattering medium of a strong external focusing field.

3. No external focusing

For completeness let us consider first the case of scattering in a material with no external focusing present. In a previous note[3] we have shown that in the gaussian limit, the beam distributions are given by

$$\begin{aligned} \langle y^2 \rangle &= \sigma_{yo}^2 + 2z \langle y_o \theta_o \rangle + \sigma_{\theta o}^2 z^2 + \frac{1}{3} \theta_C^2 z^3 \\ \langle \theta^2 \rangle &= \sigma_{\theta o}^2 + \theta_C^2 z \\ \langle y\theta \rangle &= \langle y_o \theta_o \rangle + \sigma_{\theta o}^2 z + \tfrac{1}{2} \theta_C^2 z^2 \end{aligned} \tag{10}$$

The characteristic scattering "angle" θ_C is given by

$$\theta_C = \frac{E_S}{pc\beta}\frac{1}{\sqrt{L_R}} \tag{11}$$

where E_S = 13.6 MeV, p is the particle's momentum, and L_R is the radiation length for the scattering medium. The parameter θ_C was considered a constant in the derivation of Eqs. 10. If these expressions are substituted into Eq. 7, the geometric emittance is

$$\epsilon^2 = \left[\sigma_{yo}^2\,\sigma_{\theta o}^2 + \sigma_{yo}^2\,\theta_C^2 z + \theta_C^2 \langle y_o \theta_o \rangle z^2 + \frac{1}{3}\,\theta_C^2 \sigma_{\theta o}^2 z^3 + \frac{1}{12}\,\theta_C^4 z^4\right] \tag{12}$$

If Eqs. 10 are inserted back into Eq. 8, we find that the heating term for the case of no focussing is

$$\frac{d\epsilon}{dz} = \frac{1}{2\epsilon}\left[\sigma_{yo}^2\,\theta_C^2 + 2\theta_C^2 \langle y_o \theta_o \rangle z + \theta_C^2 \sigma_{\theta o}^2 z^2 + \frac{1}{3}\,\theta_C^4 z^3\right] \tag{13}$$

4. Constant external focusing

We consider now the case of scattering in a material in the presence of an external focusing force whose strength is determined by the constant parameter

$$\omega = \sqrt{\frac{eB}{pa}} \tag{14}$$

where e is the particle's charge, B is the focusing magnetic field, and a is the radius of the focusing channel. In the gaussian limit the expectation values of the position and angle are given by [3]

$$
\begin{aligned}
<y^2> &= \sigma_{yo}^2 \cos^2(\omega z) + \frac{\sigma_{\theta o}^2}{\omega^2} \sin^2(\omega z) + z \frac{\theta_C^2}{2\,\omega^2} - \frac{\theta_C^2}{4\,\omega^3} \sin(2\,\omega z) \\
<\theta^2> &= \sigma_{\theta o}^2 \cos^2(\omega z) + \sigma_{yo}^2\, \omega^2 \sin^2(\omega z) + z \frac{\theta_C^2}{2} + \frac{\theta_C^2}{4\,\omega} \sin(2\,\omega z) \\
<y\theta> &= \frac{\sigma_{\theta o}^2}{2\,\omega} \sin(2\,\omega z) - \frac{\sigma_{yo}^2\, \omega}{2} \sin(2\,\omega z) + \frac{\theta_C^2}{2\,\omega^2} \sin^2(\omega z)
\end{aligned}
\tag{15}
$$

In deriving Eq. 15 we have assumed that there is no initial correlation between y_o and θ_o. In other words we have assumed that the initial beam entering the focusing channel is at a waist. In this case

$$
\sigma_{\theta o}^2 = \frac{\epsilon}{\beta_\perp} \tag{16}
$$

and we have the constraint

$$
\frac{\sigma_{yo}}{\sigma_{\theta o}} = \beta_\perp = \frac{1}{\omega} \tag{17}
$$

With this constraint we can write Eq. 15 in the simplified form

$$
\begin{aligned}
<y^2> &= \sigma_{yo}^2 + z \frac{\theta_C^2}{2\,\omega^2} - \frac{\theta_C^2}{4\,\omega^3} \sin(2\,\omega z) \\
<\theta^2> &= \sigma_{\theta o}^2 + z \frac{\theta_C^2}{2} + \frac{\theta_C^2}{4\,\omega} \sin(2\,\omega z) \\
<y\theta> &= \frac{\theta_C^2}{2\,\omega^2} \sin^2(\omega z)
\end{aligned}
\tag{18}
$$

The geometric emittance is

$$
\epsilon^2 = \sigma_{yo}^2\, \sigma_{\theta o}^2 + \sigma_{yo}^2\, \theta_C^2 z + \frac{\theta_C^4}{4\,\omega^2} z^2 - \frac{\theta_C^4}{8\,\omega^4} + \frac{\theta_C^4}{8\,\omega^4} \cos(2\,\omega z) \tag{19}
$$

and the heating term can be written

$$\frac{d\epsilon}{dz} = \frac{1}{2\epsilon}\left[\sigma_{yo}^2\,\theta_C^2 + \frac{\theta_C^4}{2\,\omega^2}\,z - \frac{\theta_C^4}{4\,\omega^3}\sin(2\,\omega\,z)\right] \tag{20}$$

We can obtain Neuffer's expression for heating, Eq. 3, by keeping only the first term in Eq. 20. Neglecting the second term implies that Neuffer's expression is only valid when

$$\sigma_{yo}^2 \gg \frac{\theta_C^2\,L_{rod}}{2\,\omega^2} \tag{21}$$

where L_{rod} is the length of the cooling rod, while neglecting the third term requires

$$\sigma_{yo}^2 \gg \frac{\theta_C^2}{4\,\omega^3} \tag{22}$$

The approximations in Eqs. 21 and 22 will normally be satisfied if ω is large enough (strong focusing).

5. Effect of energy loss

We have not rigorously derived expectation values for y and θ for the case when energy loss is also present. To estimate the effects of energy loss, we make the (hopefully reasonable) assumption that the basic form of Eqs. 19 and 20 are still correct with the constants θ_C and ω replaced with variables that depend on the local value of the particle energy. The decrease in particle energy is determined from dE/dz in the material. In addition we generalize Neuffer's heating term using Eqs. 14 and 17 as

$$\frac{d\epsilon_N}{dz}(heat) = \sqrt{\frac{p(z)\,a}{e\,B}}\;\frac{E_S^2}{2\,\beta^3(z)\,E(z)\,m\,c^2\,L_R} \tag{23}$$

We compare Neuffer's Eq. 23 with the energy dependent prediction of Eqs. 19 and 20 in Fig. 1, using parameters appropriate to a proposed cooling experiment at the AGS[4].

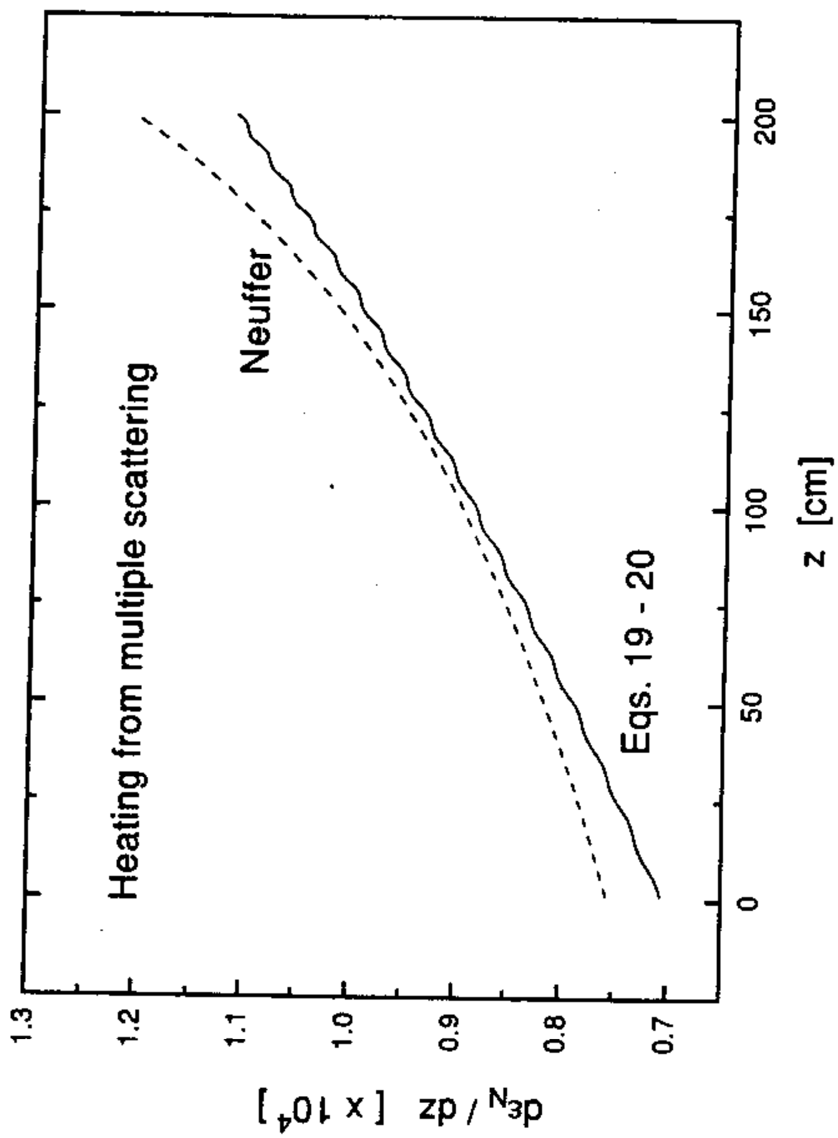

Fig. 1 Comparison of the emittance heating expression of D. Neuffer with the more exact relation derived from Eqs. 19 and 20.

We see that Neuffer's simple expression is a good approximation to the more rigorous result. Eq. 23 overestimates the amount of heating everywhere inside the cooling rod, but the deviation never exceeds about 15%.

Acknowledgements

This research was supported by the U.S. Department of Energy under Contract No. DE-AC02-76CH00016.

Notes and references

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